Output Feedback $\mathscr{H}_\infty$ Control of Constrained Linear Systems

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***Abstract*—The problem of output feedback $\mathscr{H}_\infty$ control for a class of constrained linear systems is studied in this paper. A sufficient condition for the existence of the $\mathscr{H}_\infty$ output feedback controllers is proposed. Furthermore, the solution scheme is formulated in terms of linear matrix inequalities (LMIs). Finally, an aircraft model and a quarter car active suspension model are exploited to illustrate the effectiveness of the proposed method.**

I. Introduction

The analysis and design of systems that are subject to various types of uncertainties or perturbations had always been one of the main concerns of control engineers. Feedback control is a powerful tool to deal with uncertainties. Control engineers and scholars have been trying to find out the ideal design method to guarantee the system robustness through feedback control since the 1930s. Of all kinds of robust control methods, $\mathscr{H}_\infty$ control theory is an important branch [1][2].

$\mathscr{H}_\infty$ control theory was initially proposed by Zames [3] in the early 1980s, which attempted to minimize the $\mathscr{H}_\infty$ norm of the transfer function in order to restrain disturbances to the greatest extent. The $\mathscr{H}_\infty$ norm was found to be appropriate for specifying both the level of plant uncertainty and the signal gain from disturbance inputs to error outputs in the controlled systems [4] . Here is the main idea of the $\mathscr{H}_\infty$ control : For a set of finite energy disturbance signals, design a controller to ensure that the closed-loop system is stable and the disturbance has minimal impact on the desired output simultaneously.

Finding an optimal $\mathscr{H}_\infty$ controller is often both numerically and theoretically complicated, as shown in [5]. The emergence and development of state space $\mathscr{H}_\infty$ control theory makes the design of $\mathscr{H}_\infty$ controller much easier. State space $\mathscr{H}_\infty$ control can be divided into three categories in terms of structure: $\mathscr{H}_\infty$ state feedback control, $\mathscr{H}_\infty$ output-feedback control and $\mathscr{H}_\infty$ state feedback control based on state-estimator. $\mathscr{H}_\infty$ state feedback control is investigated in [6],[7]. Since not all the states are available in practice, observer-based $\mathscr{H}_\infty$ controller and $\mathscr{H}_\infty$ output feedback controller are widely discussed. Observer-based $\mathscr{H}_\infty$ controller of networked nonlinear systems is proposed in [8],[9], where sufficient conditions for the existence of an observer-based feedback controller are derived. Robust $\mathscr{H}_\infty$ output feedback controller is investgated in [10], where the proposed schemes are related to linear matrix inequality (LMI) optimization. The problem of $\mathscr{H}_\infty$ output feedback control of commensurate Fractional Order Systems is addressed in [11], where an extension of Bounded Real Lemma is used. In these papers, LMIs are efficient ways to express $\mathscr{H}_\infty$ output feedback control problems, which offer better flexibility than analytical methods. In order to take unavailable variables into consideration and simplify the controller design, this paper choose $\mathscr{H}_\infty$ output feedback control for linear time invariant systems. As a matter of fact, most industrial systems must operate within fixed bounds and are subject to strict control limitations [12]. Thus, the input constraints and output constraints of the linear system are taken into account in this paper, which is also the characteristics of this article. In general, a problem of output feedback $\mathscr{H}_\infty$ control for a class of constrained systems is considered. Furthermore, a sufficient condition for the existence of $\mathscr{H}_\infty$ performance output feedback controllers is derived via the linear matrix inequality approach, and a design procedure is provided.

The rest of this paper is outlined as follows. First of all, the general $\mathscr{H}_\infty$ control problem of linear time invariant (LTI) system is setup in Section II. Next, both the output feedback law and solution scheme is discussed in Section III. In Section IV, examples are presented to illustrate the effectiveness of the suggested approach. Finally, some conclusions are drawn.

II. Problem Setup

Consider the following linear time invariant (LTI) system

$$\dot{x}(t) = Ax(t) + B_1 w(t) + B_2 u(t), \tag{1a}$$
$$z_1(t) = C_1 x(t) + D_{11} w(t) + D_{12} u(t), \tag{1b}$$
$$z_2(t) = C_2 x(t) + D_{21} w(t) + D_{22} u(t), \tag{1c}$$
$$y(t) = C_3 x(t) + D_{31} w(t). \tag{1d}$$

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subject to output constraints

$$|z_{2i}(t)| \le z_{2i,max}, i = 1,2,\cdots,n_{z2}, \quad t \ge 0, \tag{2}$$

where $x \in R^{n_x}$ is the state, $w \in R^{n_w}$ the persistent exogenous disturbance or uncertainty, $u \in R^{n_u}$ the control input, $z_1 \in R^{n_{z1}}$ the control output, $z_2 \in R^{n_{z2}}$ the constraint output, and $y \in R^{n_y}$ is the measurement output. It is assumed that $D_{21} = 0$ and $D_{31} = 0$, i.e., disturbance has no direct way to affect the constraint outputs and the measurement outputs. Some fundamental assumptions are stated in the following:

Assumption 1: The triple (A, B_2, C_3) is stabilizable and observable.

Assumption 2: The signal $w \in R^{n_w}$ is unknown but energy bounded, and lies in a compact set,

$$\mathscr{W} := \left\{ w \in \mathbb{R}^{n_w} \mid \int_0^\infty \|w(\tau)\|_2^2 d\tau \le w_{max} \right\}, \tag{3}$$

i.e., $w \in \mathscr{W}$ for all $t \ge 0$.

Remark 2.1: If $z_2 = u$ is chosen, the input constraint is considered in this setup.

In this paper, the dynamic output feedback control law $\mathscr{K}$ is considered,

$$\dot{\xi}(t) = A_\xi \xi(t) + B_\xi y(t), \tag{4a}$$

$$u(t) = C_\xi \xi(t) + D_\xi y(t). \tag{4b}$$

where $\xi \in R^{n_k}$ is the state of controller and n_ξ is the dimension of controller. A_ξ, B_ξ, C_ξ and D_ξ are constant matrices of appropriate dimensions which need to be determined. Applying this output feedback controller (4) to the plant (1) will present the closed-loop system

$$\dot{x}_{cl}(t) = A_{cl} x_{cl}(t) + B_{cl} w(t), \tag{5a}$$

$$z_1(t) = C_{cl,1} x_{cl}(t) + D_{cl,1} w(t), \tag{5b}$$

$$z_2(t) = C_{cl,2} x_{cl}(t) + D_{cl,2} w(t), \tag{5c}$$

where $x_{cl} = \begin{bmatrix} x \\ \xi \end{bmatrix}$, $A_{cl} = \begin{bmatrix} A + B_2 D_\xi C_3 & B_2 C_\xi \\ B_\xi C_3 & A_\xi \end{bmatrix}$, $B_{cl} = \begin{bmatrix} B_1 + B_2 D_\xi D_{31} \\ B_\xi D_{31} \end{bmatrix}$, $C_{cl,1} = \begin{bmatrix} C_1 + D_{12} D_\xi C_3 \\ D_{12} C_\xi \end{bmatrix}^T$, $D_{cl,1} = \begin{bmatrix} D_{11} + D_{12} D_\xi D_{31} \end{bmatrix}$, $C_{cl,2} = \begin{bmatrix} C_2 + D_{22} D_\xi C_3 & D_{22} C_\xi \end{bmatrix}$, $D_{cl,2} = \begin{bmatrix} D_{21} + D_{22} D_\xi D_{31} \end{bmatrix}$.

Our objective is to design a dynamic output feedback controller of the form (4) such that the closed-loop system is internally stable, the $\mathscr{H}_\infty$ performance from the disturbance w to the performance output z_1 is minimized, and the output constraints z_2 are satisfied.

III. Output Feedback $\mathscr{H}_\infty$ Control

In Section II, an $\mathscr{H}_\infty$ output feedback control problem of the general constraint LTI system was setup. In this section, $\mathscr{H}_\infty$ output feedback control law is proposed as well as the corresponding proof procedure first. Then solution scheme is provided. The $\mathscr{H}_\infty$ output feedback controller for a constraint LTI system can be designed according to it.

A. Output Feedback Law

For a given scalar $\gamma > 0$, the $\mathscr{H}_\infty$ performance from $w(t)$ to $z_1(t)$ is less than γ, if there exists a matrix such that $X_{cl} = X_{cl}^T > 0$ satisfying the following LMI [13]

$$\begin{bmatrix} A_{cl}^T X_{cl} + X_{cl} A_{cl} & X_{cl} B_{cl} & C_{cl,1}^T \\ * & -\gamma I & D_{cl,1}^T \\ * & * & -\gamma I \end{bmatrix} \le 0, \tag{6}$$

where $*$ represents the transpose of the corresponding element.

Denote $V(x_{cl}) := x_{cl}^T X_{cl} x_{cl}$. The feasibility of (6) guarantees that for the closed-loop system (5) [14]

$$\frac{d}{dt} V(x_{cl}(t)) + \|z_1(t)\|^2 - \gamma^2 \|w(t)\|^2 \le 0. \tag{7}$$

Integrating (7) from time instant 0 to $t \ge 0$, it is achieved that

$$V(x_{cl}(t)) + \int_0^t \|z_1(\tau)\|^2 d\tau \le \gamma^2 \int_0^t \|w(\tau)\|^2 d\tau + V(x_{cl}(0)). \tag{8}$$

In terms of $w \in \mathscr{W}$, (8) implies that the state trajectory stays in the ellipsoid

$$\Omega(X_{cl}, \alpha) := \Big\{ x_{cl} \in R^{n_x} \mid V(x_{cl}) \le \alpha, \quad \alpha := \gamma^2 w_{max} + V(x_{cl}(0)) \Big\}. \tag{9}$$

That is, the ellipsoid $\Omega(X_{cl}, \alpha)$ contains the set of reachable states of the closed-loop system. The next lemma is to guarantee the constraint satisfaction.

Lemma 1: Suppose that there exists a symmetric positive definite matrix X_{cl} such that the matrix inequality (6) holds for all admissible disturbance, and $X_{cl} \in \Omega(X_{cl}, \alpha)$. If

$$\begin{bmatrix} \frac{1}{\alpha} Z & C_{cl,2} \\ C_{cl,2}^T & X_{cl} \end{bmatrix} \ge 0, \tag{10}$$

then the output feedback control law $\mathscr{K}$ guarantees the satisfaction of the constraint (2), where $Z_{ii} \le z_{2i,max}^2$, for $i = 1,2,\cdots,n_{z2}$.

Proof: According to Cauchy-Schwarz inequality [15], it follows:

$$\begin{aligned} \max_{t\ge0} |z_i(t)|^2 &= \max_{t\ge0} |C_{2i} x_{cl}(t)|^2 \\ &\le \max_{x\in\Omega} |C_{2i} x_{cl}(t)|^2 \\ &= \max_{x\in\Omega} |C_{2i} X_{cl}^{-\frac{1}{2}} X_{cl}^{\frac{1}{2}} x_{cl}(t)|^2 \\ &\le \|(C_2 X_{cl}^{-\frac{1}{2}})_i\|_2^2 \cdot \|X_{cl}^{\frac{1}{2}} x_{cl}\|_2^2 \\ &\le \alpha (C_2 X_{cl}^{-1} C_2^T)_{ii}, \end{aligned}$$

where the fifth inequality is due to $X_{cl} \in \Omega(X_{cl}, \alpha)$. *In terms of the Schur Complement Theorem [16], the inequality (10) is equivalent to*

$$\alpha (C_2 X_{cl}^{-1} C_2^T)_{ii} \le \max_{t\ge0} |z_i(t)|^2 = z_{2i,max}^2$$

□

B. Solution Scheme

In the sequel, a method named changing variables [17] is used to reformulate (6) and (10) as LMIs. Partition X_{cl} and X_{cl}^{-1} as

$$X_{cl} = \begin{bmatrix} Y & N \\ N^T & * \end{bmatrix}, \qquad X_{cl}^{-1} = \begin{bmatrix} X & M \\ M^T & * \end{bmatrix}, \tag{11}$$

where $*$ represents that the block is arbitrary, both X and Y are symmetric and of the same size as A. In terms of $X_{cl}X_{cl}^{-1} = I$, $X_{cl}\begin{bmatrix} X \\ M^T \end{bmatrix} = \begin{bmatrix} I \\ 0 \end{bmatrix}$ is infered, which leads to

$$X_{cl}\Pi_1 = \Pi_2$$

with $\Pi_1 = \begin{bmatrix} X & I \\ M^T & 0 \end{bmatrix}$, $\Pi_2 = \begin{bmatrix} I & Y \\ 0 & N^T \end{bmatrix}$.

Let us define the changes of controller variables as follows,

$$\begin{aligned} \hat{A} &:= NA_\xi M^T + NB_\xi C_3 X + YB_2 C_\xi M^T \\ &\quad + Y(A + B_2 D_\xi C_3)X \end{aligned}, \tag{12a}$$

$$\hat{B} := NB_\xi + YB_2 D_\xi, \tag{12b}$$

$$\hat{C} := C_\xi M^T + D_\xi C_3 X, \tag{12c}$$

$$\hat{D} := D_\xi. \tag{12d}$$

Suppose that M and N have full row rank, and if $\hat{A}, \hat{B}, \hat{C}, \hat{D}$ and X, Y are given, controller matrices A_ξ, B_ξ, C_ξ and D_ξ can be derived such that (12) is satisfied. Furhermore, if M and N are square and invertible, then A_ξ, B_ξ, C_ξ and D_ξ are unique [2]. For the full order design, one can always assume that M and N have full row rank. Hence the variables $A_\xi, B_\xi, C_\xi, D_\xi$ can be replaced by $\hat{A}, \hat{B}, \hat{C}, \hat{D}$ without loss of generality.

If both a congruence transformation with $diag(\Pi_1, I, I)$ on (6) and a congruence transformation with $diag(I, \Pi_1)$ on (10) are performed, the following inequalities can be achieved

$$\begin{bmatrix} S_0 & S_1 & B_1 & (C_1 X + D_{12}\hat{C})^T \\ \star & S_2 & YB_1 & (C_1 X + D_{12}\hat{D}C_2)^T \\ \star & \star & -I & (D_{11} + D_{12}\hat{D}D_{31})^T \\ \star & \star & \star & -\gamma^2 I \end{bmatrix} \leq 0 \tag{13}$$

and

$$\begin{bmatrix} \frac{1}{\alpha}Z & M_0 & M_1 \\ \star & X & I \\ \star & \star & Y \end{bmatrix} > 0 \ with\ Z_{ii} \leq z_{2i,max}^2 , \tag{14}$$

where $\star$ replaces blocks that are readily inferred by symmetry, $S_0 = AX + XA^T + B_2\hat{C} + (B_2\hat{C})^T$, $S_1 = \hat{A}^T + A + B_2\hat{D}C_3$, $S_2 = A^T Y + YA + \hat{B}C_2 + (\hat{B}C_2)^T$, $M_0 = C_2 X + D_{22}\hat{C}$, $M_1 = C_2 + D_{22}\hat{D}C_3$.

Clearly, the above inequalities are LMIs in X, Y, and $(\hat{A}, \hat{B}, \hat{C}, \hat{D})$ for a fixed α. The semi-definite programming

$$\min_{X>0, Y>0, \hat{A}, \hat{B}, \hat{C}, \hat{D}} \gamma \quad s.t.\ LMIs\ (13), (14) \tag{15}$$

is convex and numerically tractable. Now, it is the place to state the main result of this paper.

Theorem 1: Suppose that there exists an optimal solution X, Y, and $(\hat{A}, \hat{B}, \hat{C}, \hat{D})$ to (15). then, the output feedback controller (4) guarantees

1) a disturbance attenuation level γ^* for all energy bounded disturbances;
2) satisfaction of the time-domain hard constraints (2), if the disturbance energy satisfies (3).

Proof: 1). The existence of matrices $X > 0, Y > 0$, and $\hat{A}, \hat{B}, \hat{C}, \hat{D}$ satisfying LMI (13) guarantees the dissipation inequality (7). Suppose that $x(0) = 0$, then (7) is sufficient for

$$\frac{\|z_1\|}{\|w\|} \leq \gamma^*$$

2). Moreover, if the disturbance energy satisfies (3), then the closed-loop trajectory is contained in $\Omega(X_{cl}, \alpha)$. This in turn guarantees that the feasibility of LMIs (14) implies the satisfaction of the hard constraints. □

Remark 3.1: Since (A,B) is stabilizable, an $\alpha > 0$ can be found such that the LMI optimization problem is feasible [17].

Suppose that the LMI optimization problem has an optimal solution $(\gamma^*, X^*, Y^*, \hat{A}^*, \hat{B}^*, \hat{C}^*, \hat{D}^*)$, the construction proceeds of the controllers as follows: find nonsingular matrices M and N to satisfy $MN^T = I - XY$ and define the controller by [14]

$$D_\xi^* := \hat{D}, \tag{16a}$$

$$C_\xi^* := (\hat{C}^* - D_\xi C_3 X)M^{-T}, \tag{16b}$$

$$B_\xi^* := N^{-1}(\hat{B}^* - YB_2 D_\xi), \tag{16c}$$

$$\begin{aligned} A_\xi^* &:= N^{-1}(\hat{A}^* - NB_\xi C_3 X - YB_2 C_\xi M^T \\ &\quad - YAX - YB_2 D_\xi C_3 X)M^{-T}. \end{aligned} \tag{16d}$$

IV. Simulation

In this section, two design examples are presented to illustrate the effectiveness of the proposed controller design method. A relatively simple example is considered in the first simulation, which is about the lateral direction control of an aircraft. In the second simulation, the active suspension based on a quarter-car is fully discussed. For comparison, the $\mathcal{H}_\infty$ /generalized H_2 control method introduced in [14] is also employed to demonstrate the effectiveness of the proposed method.

A. Application to Flight Control

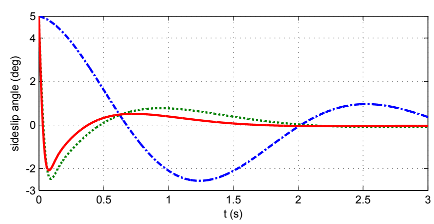
This example is taken from the linearized small perturbation model of an aircraft [18], its lateral direction state equation is described in the form of (1), where $x = [\beta\ \omega_x\ \omega_z\ \gamma]$, β is sideslip angle, ω_x is roll angular velocity, ω_z is yaw rate and γ is roll angle, $u = [\delta_a\ \delta_z]$, which stand for aileron angle and rudder angle. Parameter matrices are written as follow:

$$A=\begin{bmatrix} -0.2289 & 0.0555 & 0.9933 & 0.2519\\ -45.3705 & -19.6189 & -11.144 & 0\\ -4.6579 & 0.3565 & -0.6407 & 0\\ 0 & 1 & -0.0566 & 0\end{bmatrix},$$
$$B_1=\begin{bmatrix} -0.2289 & -45.3705 & -4.6579 & 0\end{bmatrix},$$
$$B_2=\begin{bmatrix} -0.0215 & -49.2930 & 2.9839 & 0\\ -0.0727 & -4.1173 & -6.2686 & 0\end{bmatrix},$$
$$C_1=\begin{bmatrix}1&0&0&0\\0&0&0&1\end{bmatrix}, C_2=\begin{bmatrix}0&1&0&0\\0&0&1&0\\0&0&0&0\\0&0&0&0\end{bmatrix}, D_{12}=\begin{bmatrix}0&0\\0&0\\1&0\\0&1\end{bmatrix},$$

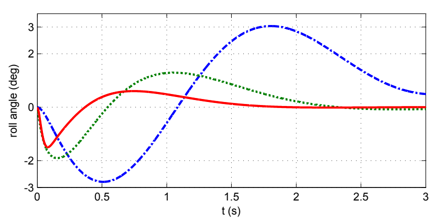
$D_{11}, D_{21}, D_{22}, D_{31}$ are zero matrices with appropriate dimensions. Initial state condition is $x_0 = [5\ 0\ 0\ 0]$. The additional force caused by air disturbance can be described as

$$\omega(t)=\begin{cases} 0.2\ , & 1s \le t \le 2s\\ 0\ , & otherwise\end{cases} \tag{17}$$

The $\mathscr{H}_\infty$ output feedback controller is obtained as proposed scheme. The simulation results are shown in Figure 1, where $\mathscr{H}_\infty$ /generalized H_2 control is given as a comparison [c.f.[14]]. The achieved $\mathscr{H}_\infty$ performance of the methods are 0.1707 and 0.1802, respectively. The simulation results show that the proposed controller effectively reduces the sideslip angle and yaw angle in yaw and roll flight condition, and can resist the influence of air disturbances.



(a) Sideslip Angle



(b) Roll Angle

Figure 1: Air disturbance responses : output feedback $\mathscr{H}_\infty$ control (-), output feedback $\mathscr{H}_\infty$ /generalized H_2 (...) , without any controller (-.-)

B. Application to 1/4 Active Suspensions

1) ***2-DOF Quarter-Car Model and Control Problem Formation:*** 2-DOF quarter-car models are widely used in suspension analysis and design, because they capture major characteristics of a real suspension system. A generalized quarter-car suspension model is shown in Figure 2, where m_s and m_u stand for the sprung mass and unsprung mass; k_s and c_s are stiffness and damping of the suspension system, respectively. In addition, (k_s, c_s) consists of the so-called passive suspension; k_u represents the tire stiffness; $x_s - x_u$ denotes the suspension stroke and x_g is vertical ground displacement caused by road unevenness. Moreover, u_f is the active control force provided by a hydraulic actuator. The model parameters are given in Table 1 for the controller design. Based on this suspension model, the linearized dynamic equations of the sprung and unsprung mass can be established [6]:

$$m_s\ddot{x}_s + k_s(x_s - x_u) + x_s(\dot{x}_s - \dot{x}_s) = u_f \tag{18a}$$
$$m_u\ddot{x}_s - k_s(x_s - x_u) - x_s(\dot{x}_s - \dot{x}_s) + k_u(x_u - x_g) = u_f \tag{18b}$$

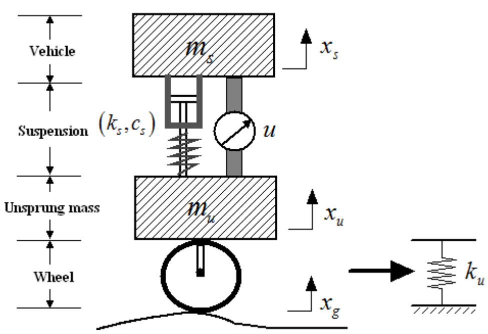


Figure 2: Quarter-car model

Define a set of state variables $x_1 = x_s - x_u, x_2 = \dot{x}_s, x_3 = x_u - x_g, x_4 = \dot{x}_u$, the state description of the car motion can be obtained as

$$\begin{aligned}\dot{x}(t) &= \begin{bmatrix} 0 & 1 & 0 & -1\\ -\frac{k_s}{m_s} & -\frac{c_s}{m_s} & 0 & \frac{c_s}{m_s}\\ 0 & 0 & 0 & -1\\ \frac{k_s}{m_u} & \frac{c_s}{m_u} & -\frac{k_u}{m_u} & -\frac{c_s}{m_u}\end{bmatrix} x(t)\\ &+\begin{bmatrix}0\\0\\-1\\0\end{bmatrix} w(t) + \begin{bmatrix}0\\ \frac{u_s}{m_s}\\ 0\\ -\frac{u_s}{m_u}\end{bmatrix} u(t)\ ,\end{aligned} \tag{19}$$

where $w = \dot{x}_g$ denotes the disturbance input caused by road roughness, u_s is the maximum active force and $u = u_f/u_s$ is the normoalized active force.

The ride comfort is the most basic concerning in the design of a suspension system which is quantified by the body acceleration in the vertical direction. It is reasonable to choose the $\mathscr{H}_\infty$ norm as performance measure, since the value of $\mathscr{H}_\infty$ norm actually generates an upper bound on the root mean square (RMS) gain [6]. The body acceleration $\ddot{x}_s$ is chosen as the performance output in the form of

$$z_1(t) = \begin{bmatrix} -\frac{k_s}{m_s} & -\frac{c_s}{m_s} & 0 & \frac{c_s}{m_s}\end{bmatrix} x(t) + \frac{u_s}{m_s} u(t) \tag{20}$$

In addition, road holding, which has influence on vehicle handling stability, is another key suspension performance.

Table I: Parameters in a Quarter Car Model

Model parameters	Values
sprung mass (m_s)	320kg
suspension stiffness (k_s)	18kN/m
suspension damping rate (c_s)	$1kN \cdot s \cdot m^{-1}$
unsprung mass (m_u)	40kg
tire stiffness (k_u)	200kN/m
maximum active force (u_s)	1000N
maximum suspension deflection (S_{max})	0.08m

In order to ensure a firm uninterrupted contact of wheels to road, the dynamic tire load should not exceed the static tire load

$$k_u(x_u(t) - x_g(t)) < (m_s + m_u)g. \tag{21}$$

Due to the mechanical structure, the suspension stroke limitation as follows should be considered so that the suspension stoke will not exceed the allowable maximum

$$|\, x_s(t) - x_u(t) \,| \leq S_{max}, \tag{22}$$

where S_{max} is the maximum suspension deflection. Moreover, the actuator saturation should be taken into account, thus the normalized active control force is bounded

$$|\, u(t) \,| \leq 1. \tag{23}$$

Given the above discussions, suspension stroke $x_s - x_u$, relative dynamic tire load $\frac{k_u(x_u - x_g)}{(m_s+m_u)g}$ and active force u can be chosen as the constraint output $z_2(t)$,

$$\begin{aligned} z_2(t) &= \begin{bmatrix} \frac{x_s - x_u}{S_{max}} \\ \frac{k_u(x_u - x_g)}{(m_s+m_u)g} \\ u \end{bmatrix} \\ &= \begin{bmatrix} \frac{1}{S_{max}} & 0 & 0 & 0 \\ 0 & 0 & \frac{k_u}{(m_s+m_u)g} & 0 \\ 0 & 0 & 0 & 0 \end{bmatrix} x(t) + \begin{bmatrix} 0 \\ 0 \\ 1 \end{bmatrix} u(t). \end{aligned} \tag{24}$$

In summary, the objective is to find an internally stabilizing linear dynamic output feedback controller (4) such that:

1) The performance output $z_1(t)$ is minimized in order to improve the ride comfort;
2) The absolute value of each element of the constraint output $z_2(t)$ is less than one so as to satisfy the corresponding time-domain hard constraints.

*2) **Response to Disturbances**:* To evaluate the effectiveness of the designed controller, a case of an isolated bump in an otherwise smooth road surface is considered. The corresponding ground displacement is given by

$$\frac{A}{2}\left(1 - \cos\frac{2\pi v}{L}t\right), \quad 0 \leq t \leq \frac{L}{v} \tag{25}$$

where A and L are the height and length of the bump respectively. As there are different road roughness, C Grade (average) road surface with $G_0 = 128 \times 10^{-6} m^3$ is considered, where G_0 stands for the road roughness coefficient. The disturbance energy of this bump can be described by the following ground velocity :

$$\dot{x}_g(t) = \begin{cases} N(0, 2\pi n_0 \sqrt{G_0 v}) + \frac{\pi v A}{L}\sin(\frac{2\pi v}{L}t), & 0 \leq t \leq \frac{L}{v} \\ N(0, 2\pi n_0 \sqrt{G_0 v}), & t > \frac{L}{v} \end{cases} \tag{26}$$

where n_0 denotes the spatial frequency and $n_0 = 0.1m^{-1}$ is the reference spatial frequency. Choose $A = 0.1m$, $L = 5m$, and the vehicle forward velocity as $v = 12.5m/s$ ($= 45km/h$), c.f. Figure 3. The output feedback controller (4) is obtained by solving the optimization problem (15). The achieved $\mathcal{H}_\infty$ performance is $\gamma^* = 9.8696$, which implies the anti-interference performance of the active suspension. Figure 4 demonstrates the responses of ground velocity, vertical accelerations, suspension strokes, relative dynamic tire load and active force for the passive and active suspensions system. In order to evaluate the performance of the designed controller, a $\mathcal{H}_\infty$ /generalized H_2 output feedback controller (cf.[14]) is proposed here as a comparison. The achieved $\mathcal{H}_\infty$ performance $\mathcal{H}_\infty$ /generalized H_2 output feedback is 10.3825. The results of the simulation show that the system with $\mathcal{H}_\infty$ output feedback controller has better performance than $\mathcal{H}_\infty$ /generalized H_2 output feedback control. Both the active suspensions have improved ride comfort and relative dynamic tire load significantly compared to the passive suspension, while the suspension stroke and actuator force are under its prespecified bounds.

V. Summary

An output feedback $\mathcal{H}_\infty$ control for constrained LTI systems has been investigated in this paper. A sufficient condition for the existence of guaranteed $\mathcal{H}_\infty$ performance output feedback controllers is derived via the linear matrix inequality approach. The $\mathcal{H}_\infty$ output feedback controller can be easily obtained according to the solution of a convex optimization problem. The lateral direction control of an aircraft and active suspension control problem were shown in order to demonstrate its effectiveness. Results both of the two simulations showed that the proposed method could effectively restrain disturbance to some extent.

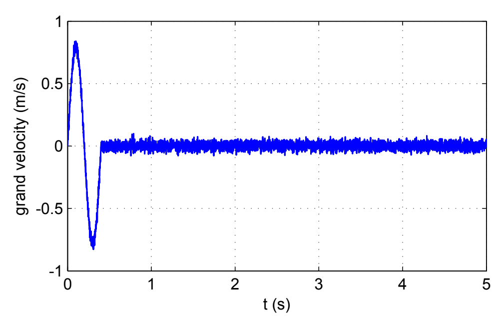
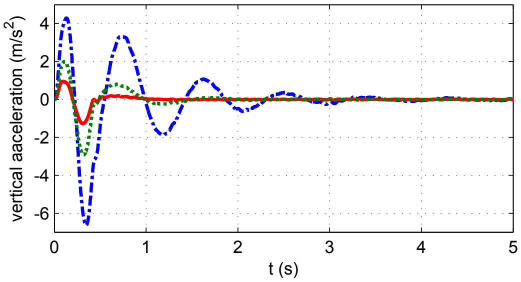
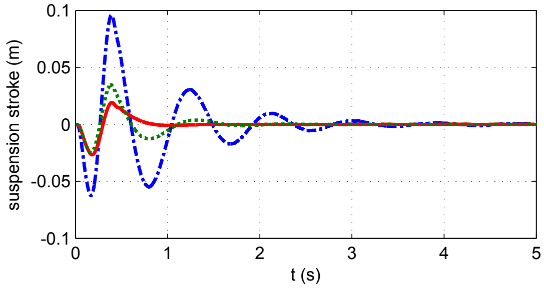


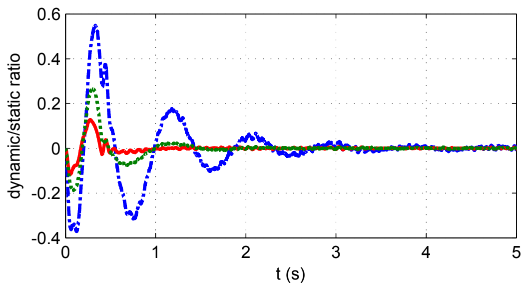
Figure 3: Ground Velocity



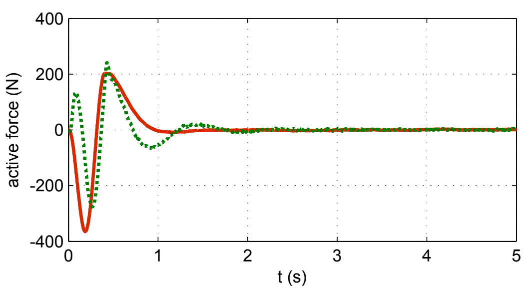
(a) Vertical Acceleration



(b) Suspension stroke



(c) Dynamic/Static Load Ratio



(d) Active Force

Figure 4: Bump responses: output feedback $\mathscr{H}_\infty$ control (-), output feedback $\mathscr{H}_\infty$ /generalized H_2 (...) , passive (-.-)

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